ON THE EXTREMAL PROPERTIES OF ENTROPY ON A CRITICAL STREAMLINE(*)

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We shall consider a three-dimensional flow past a smooth body, of a turbulent stream of an ideal gas. We assume that a critical streamline exists which branches in all directions at the isolated stagnation point O on the surface of the body, so that in the sufficiently small neighborhood of O, the whole surface is covered with streamlines issuing from this point. We introduce a coordinate system connected with the tubes of flow near both the critical streamline and the surface of the body.

Let W be a plane normal to the critical streamline at the point A, different from O. We shall produce in this plane a family of circles with a common center at A, and we shall assume that stream tubes passing through these circles form a coordinate family u_2. On the plane W we have $u_2 = r$, where r is the radius of a circle giving rise to u_2. When $r = 0$, then the stream tube u_2 consists of the surface of the body together with the critical streamline.

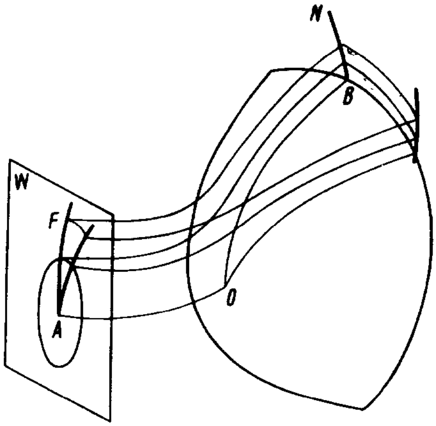

Fig. 1

Let us draw on the surface of the body in sufficiently small neighborhood of O, a closed contour L intersecting each streamline issuing from O once only (point O is situated on the surface of the body, inside L). Through each point of the contour L, we shall draw a curve N touching neither the streamlines, nor the surface of the body. We shall call the stream tubes passing through the family of curves N, a coordinate family u_3.

The only requirement imposed on u_1 (which is not a family of stream tubes) is that one of the surfaces of this family, namely the one which passes through A should, at this point, make contact with the plane W, and that this point of contact is of sufficiently high order.

Let the constructed coordinate system be triply orthogonal at some arbitrary point B on the surface of the body. We shall denote by F a line, $u_3 =$ const, on W. All F intersect at the point A, forming a knot, since each surface u_3 is continuous and contains a streamline issuing from the critical streamline at the point O. It is easy to establish that the type of this knot does not depend on the choice of curves N defining the directions of u_3. For definiteness we shall assume that all N are orthogonal to the surface of the body.

Derivatives in the directions of the lines of intersection of surfaces u_2 and u_3, u_1 and u_3, u_1 and u_2 shall be denoted by

$$\frac{\partial}{\partial s_i} = \frac{1}{h_i}\frac{\partial}{\partial u_i}, \qquad h_i = (x_{u_i}^2 + y_{u_i}^2 + z_{u_i}^2)^{1/2} \qquad (i = 1, 2, 3)$$

*) The author had learned that a related problem was investigated by M.D. Ladyzhenskii.

where h_i are Lamé coefficients.

Equation of continuity which assumes the form of a Law of Conservation of Mass for an elementary stream tube formed by the surfaces u_2, $u_2 + du_2$, u_3 and $u_3 + du_3$, has the form

$$q(\lambda)\, h_2 h_3 \,|\sin\beta\cos\gamma| = G(u_2, u_3), \quad q(\lambda) = \lambda\,[{}^1/_2\,(k+1) - {}^1/_2\,\lambda^2(k-1)]^{1/(k-1)} \quad (1)$$

Here λ is the velocity coefficient, γ is an angle between the normal to u_1 and the velocity vector, β is an angle formed between the lines of intersection of u_2 and u_3 with u_1 while $G(u_2, u_3)$ is an arbitrary function which characterises the outflow of gas through an elementary stream tube and which depends on the choice of u_2 and u_3.

Forming a scalar product of Eq. $(\mathbf{V}\cdot\nabla)\mathbf{V} + \rho^{-1}\nabla p = 0$ with a unit vector $\mathbf{n}$ normal to u_2 and with a unit vector $\boldsymbol{\tau}$ of the normal to a streamline and tangent to u_2 we obtain, using the Bernoulli's equation and the condition of isentropy.

$$\varkappa_n = -\frac{\partial\ln\lambda}{\partial n} - \frac{1}{kRM^2}\frac{\partial S}{\partial n}, \qquad \varkappa_\tau = -\frac{\partial\ln\lambda}{\partial\tau} - \frac{1}{kRM^2}\frac{\partial S}{\partial\tau} \quad (2)$$

where $\varkappa_n$ and $\varkappa_\tau$ are the normal and geodesic curvature of a streamline on u_2, $\partial/\partial n$ and $\partial/\partial\tau$ denote partial derivatives in the $\mathbf{n}$ and $\boldsymbol{\tau}$ direction and k is the adiabatic index. While deriving (2), we have assumed without any loss of generality that the stagnation temperature was constant throughout the flow.

At a point B on a body surface where the coordinate system is tri-orthogonal, Eqs. (2) become

$$\varkappa_n = -\frac{\partial\ln\lambda}{\partial s_2} - \frac{1}{kRM^2}\frac{\partial S}{\partial s_2}, \qquad \varkappa_\tau = -\frac{\partial\ln\lambda}{\partial s_3} \quad (3)$$

since by a previous assumption of existence of a critical streamline, entropy is constant on the body surface. As $S = S(u_2, u_3)$, the derivative $\partial S/\partial s_2$ can, at the point B, be written as $\partial S/\partial s_2$ in the direction of a corresponding line F at A. Using (1) we obtain

$$\left.\frac{\partial S}{\partial s_2}\right|_B = \left.\frac{\partial S}{\partial s_2}\right|_A \frac{h_2|_A}{h_2|_B} = \left.\frac{\partial S}{\partial s_2}\right|_A \frac{[q(\lambda)\,h_3]_B}{[q(\lambda)\,h_3\sin\beta]_A} \quad (4)$$

The fact that a knot is formed by the curves F at A implies, that on W, Lame's coefficients $h_3 \to 0$ as $u_2 \to 0$. At the same time u_3 can always be relabelled so as to have $h_3 \neq 0$ on the surface of the body.

We can assume without loss of generality that $\lambda \neq 0$ at B. Since the normal curvature of streamlines on the surface of a smooth body is limited, it follows from (3) and (4) that the necessary condition for a derivative $\partial\lambda/\partial n$ on the surface of a body to be bounded is that the derivative $\partial S/\partial s_2$ in the direction F becomes equal to zero at A.

Since B is arbitrary, the derivative $\partial S/\partial s_2$ should become zero on translation along any F towards the point A. If the node of lines F resembles (or partially resembles) a star, then the necessary condition for $\partial\lambda/\partial n$ to be bounded on the surface of the body is, that the necessary conditions of existence of an extremum of entropy hold at the point A. The same conclusion is reached when the node is not star-shaped, provided that all the lines F have no point of contact with the surface $S =$ const at A.

We shall now assume that the node of lines F at A is not star-shaped and that a certain direction from this node (all lines F belonging to some pencil of lines, are tangent to this direction) touches the surface $S =$ const passing through A. Let us now introduce, on the plane W, a Cartesian coordinate system xAy in which the direction of the Ax-axis coincides with the direction described in the last sentence.

Suppose a family of curves F on the plane W is given by $F = F(x, y)$. On translation along any curve F which does not coincide with the Ax-axis in the direction towards A, we have

$$F_x \to \infty, \quad F_y \to \infty, \quad dy/dx = -F_x/F_y \to 0 \quad (5)$$

where x and y are coordinates of a point on F.

Lame coefficient h_3 at a point C on F and sufficiently near to A, is given by

$$h_3 = \frac{1}{|\nabla F|\sin\beta} = \frac{1}{F_y}\left(1 + \frac{y^2}{x^2}\right)^{1/2}\left(1 - \frac{yF_x}{xF_y}\right)^{-1/2} \quad (6)$$

where β is the angle between the curves $F =$ const and $u_2 =$ const, at the point C.

In the sufficiently near vicinity of A on $F =$ const we shall have $y/x \to 0$ as $C \to A$, consequently, when point C moves along $F =$ const towards A, then the Lame coefficient h_3 decreases as $1/F_y$.

The derivative $\partial S/\partial s_2$ in the direction of F at C, can be written as

$$\frac{\partial S}{\partial s_2} = \frac{-S_x F_y + S_y F_x}{\sqrt{F_x^2 + F_y^2}} \tag{7}$$

Let the derivatives S_x and S_y be continuous in some neighborhood of A, and

$$S_x = O(r^\delta), \quad S_y = S_y|_A + O(r^\varepsilon), \quad r = (x^2 + y^2)^{1/2} \quad (\delta > 0, \varepsilon > 0) \tag{8}$$

Substituting (8) into (7) we obtain

$$\frac{\partial S}{\partial s_2} = \frac{S_y|_A F_x + F_y O(r^\delta) + F_x O(r^\varepsilon)}{\sqrt{F_x^2 + F_y^2}} \tag{9}$$

The derivative $\partial S/\partial s_2$ at the point B is then given, in accordance with (5), (6) and (9), by

$$\frac{1}{h_3|_B} \frac{\partial S}{\partial s_2}\bigg|_B = \frac{q(\lambda_B)}{q(\lambda_A)} \lim_{C\to A} \left[\frac{1}{h_3 \sin\beta} \frac{\partial S}{\partial s_2}\right]_C = \frac{q(\lambda_B)}{q(\lambda_A)} \lim_{C\to A} [F_y O(r^\delta) + S_y|_A F_x]_C$$

We easily see that the expression in square brackets can be bounded only on an isolated curve $F=$ const. Therefore, if $S_y \neq 0$ at A, then, on approaching B along the surface u_3 the derivative $\partial S/\partial s_2 \to \infty$ at least as fast as $F_x|_C$ when approaching point A along the corresponding curve F.

Thus we have in fact shown that, if a segment exists near O on the surface of the body which includes more than one streamline and on which $\partial\lambda/\partial n$ is bounded, then necessary conditions for existence of an extremal value of entropy hold on the critical streamline.

If, on the other hand, these conditions are not fulfilled on the critical streamline, then the derivatives $\partial\lambda/\partial n$ and $\partial S/\partial n$ become infinite almost everywhere on the surface of the body with the exception of isolated streamlines on which $\partial S/\partial n$ and $\partial\lambda/\partial n$ change their sign. This means that the surface of the body is an envelope of surfaces $\lambda=$ const almost everywhere.

Indeed, conditions limiting the velocity $\lambda \leqslant \sqrt{(k+1)/(k-1)}$ implies that points at which $\partial\lambda/\partial s_1$ become infinite in the direction of the streamlines, are isolated on the streamline. Condition limiting the angles of inclination of the velocity vector imply, that points at which the geodesic curvature of streamlines on the surface of the body becomes infinite, are also isolated on a streamline. Hence it follows from (3) that points at which $\partial\lambda/\partial s_3 = \infty$ are also isolates on a streamline.

The manner in which the surfaces $\lambda=$ const touch the surface of the body, will be dependent on the sign of $\partial S/\partial n$ on the body surface: either the surface $\lambda=$ const is tangent to the direction of a streamline approaching O, or it is tangent to the direction of a streamline departing from O. On the streamlines along which $\partial S/\partial n$ changes its sign, the character of the contact changes occurs; the surfaces $\lambda=$ const are not smooth along these streamlines.

In conclusion we note that the fulfillment on the critical streamline of the necessary conditions of existence of extremal values of entropy is not, in general, a sufficient condition for the boundedness of the derivatives $\partial S/\partial n$ and $\partial\lambda/\partial n$ on the surface of the body. As an example, we shall quote the case of an axisymmetric, turbulent, subsonic flow past a body. In this flow entropy near the critical streamline at an infinite distance from the body (where the flow is rectilinear), obeys the law $S = Y^\alpha$ where Y is the ordinate of a streamline at infinity and $1 < \alpha < 2$. We can easily see that in this case the derivative $\partial S/\partial s_2$ tends to infinity like $Y^{\alpha-2}$ as it approaches the body (i.e. as $Y \to 0$).

Translated by L.K.